Studies on various notions of *ip*-convexity on directed acyclic graphs and its relationship with directed graphs

Krishnadas K[1], Prasanth G. Narasimha-Shenoi[2], Bijo S.Anand[3], and Manoj Changat[4]

[1]Department of Mathematics, Panampilly Memorial Government College, Chalakudy, Thrissur, Kerala-680722.
[2]Department of Mathematics, Government College Chittur, Palakkad, Kerala, India- 678104.
[3]Department of Mathematics, Sree Narayana College, Punalur Kollam, Kerala, India- 691305.
[4]Department of Futures Studies, University of Kerala, Thiruvananthapuram, Kerala, India- 695581.

April 30, 2026

Abstract

Let $D = (V, A)$ be a digraph and X a subset of V. Then, X is *incoming - path convex* (*ip-convex*) if, for any vertex $v \in X$, every incoming path of v is completely contained in X. Here we investigate the convexity number, rank, interval number, and the convexity invariants such as Carathèodory number, Helly number, Radon number, and exchange number with respect to the *ip-convex*.

1 Introduction

A convexity on a set is a family of subsets which is stable for intersection and nested unions.The notion of *convexity* finds a large number of applications in several fields. There are several convexity spaces defined in the set of vertices of a graph. *Directed graphs* (or short *digraphs*) and *digraph convexity* [1, 3, 6, 8] have immense applications in almost all areas of science. In [5] *ip-convexity* in *digraphs* was introduced and some results are proved in *ip-convex hulls* and in *ip-hull sets*. Another important idea that is used in various fields of Mathematics is the Exchange properties. The invariant exchange number for a convexity space e was introduced by Sierksma [7] in 1975. The relationship between the exchange number and the classical convexity invariants, namely the Carathèodory, Helly, and Radon numbers, has been studied by these authors. The relation $e \leq c + 1$ is proved by Sierksma in [7]. Our aim is to study convexity notions such as the convexity number[4], rank, interval number, and convexity invariants such as Carathèodory number, Helly number, Radon number and exchange number with respect to *ip-convex*.

2 Preliminaries

A family $\mathcal{C}$ of subsets of a finite set X is a convexity on X if $\emptyset \in \mathcal{C}, X \in \mathcal{C}$, and $\mathcal{C}$ is closed under intersections. The elements of a convexity $\mathcal{C}$ are called *convex sets* and the ordered pair $(X, \mathcal{C})$ is a *convexity space*. The *convex hull* of a subset $S \subseteq X$, denoted by $< S >$ is the intersection of all convex sets containing S. A subset of V is a *hull set* if its convex hull is equal to V, and the *hull number* is the minimum cardinality of hull sets.

In [5], a set X of vertices of a digraph D is *incoming - path convex* (*ip-convex*) if, for any vertex $v \in X$, every incoming path of v is completely contained in X. Let $\mathcal{C}$ be the collection of *ip*-convex sets in a digraph D on V. We call the pair $(V, \mathcal{C})$ the *ip-convexity space* of D.

The *ip-convexity number* is the maximum cardinality of a proper *ip*-convex set in $\mathcal{C}$. The Cardinality of a maximum *ip*-convexly independent set is known as the *ip-rank*. The *ip-interval number* is the cardinality of a smallest *ip*-interval set X of V. A convex structure X gives rise to the following numbers $h(X), r(X), c(X), e(X), \in \{0, 1, \ldots, \infty\}$ determined by the following prescription. $h(X)$ is the maximum cardinality of a *H-independent set*, $c(X)$ is the maximum cardinality of a *C-independent set*, $e(X)$ is the maximum cardinality of a *E-independent set*. In [2], the *Radon number*is the smallest integer r (if it exists) such that every r-element set $S \subseteq X$ admits a Radon partition.

3 Convexity Number, Rank, and Interval Number

Theorem 1. *If $D = (V, A)$ is a DAG with n vertices, then the ip-convexity number is $n - 1$.*

Theorem 2. *If D is a digraph and $\tilde{D}$ is its superstructure, then ip-rank of D and ip-rank of $\tilde{D}$ are the same.*

Theorem 3. *If D is a digraph and $\tilde{D}$ is its superstructure, then the ip-rank of D is the cardinality of the largest set containing mutually unreachable sources and sinks in $\tilde{D}$.*

Theorem 4. *If $D = (V, A)$ is a DAG, then the ip-interval number of D is the number of its sinks.*

Theorem 5. *If $D = (V, A)$ is a digraph, then the ip-interval number of D is the number of sinks in its superstructure $\tilde{D}$.*

4 The Carathèodory, Helly, Radon, and exchange number

Theorem 6. *If $D = (V, A)$ is a directed graph, with ip-convexity $\mathcal{C}$, then the Carathèodory number is $c = 1$.*

Theorem 7. *If $D = (V, A)$ is a directed graph which is not Hamiltonian, with ip-convexity $\mathcal{C}$, then the exchange number is $e = 2$.*

Theorem 8. *If D is a digraph and $\tilde{D}$ is its superstructure, then Helly number of D and Helly number of $\tilde{D}$ are the same.*

Theorem 9. *If $D = (V, A)$ is a DAG, with ip-convexity $\mathcal{C}$, then the Helly number is the number of sources in D.*

Theorem 10. *If D is a digraph and $\tilde{D}$ is its superstructure, then the Radon number of D and the Radon number of $\tilde{D}$ are the same.*

Theorem 11. *If $D = (V, A)$ is a DAG, with ip-convexity $\mathcal{C}$, then the Radon number is the number of sources in D plus one.*

References

[1] Bijo S Anand, Manoj Changat, Prasanth G Narasimha-Shenoi, and Mary Shalet Thottungal Joseph, *Boundary-type sets of strong product of directed graphs*, Ars Mathematica Contemporanea **20** (2021), no. 2, 275–288.

[2] Jørgen Bang-Jensen and Gregory Gutin, *Classes of directed graphs*, vol. 11, Springer, 2018.

[3] Manoj Changat, Prasanth G Narasimha-Shenoi, Mary Shalet Thottungal Joseph, and Ram Kumar, *Boundary vertices of cartesian product of directed graphs*, International Journal of Applied and Computational Mathematics **5** (2019), no. 1, 1–19.

[4] Gary Chartrand, Curtiss E Wall, and Ping Zhang, *The convexity number of a graph*, Graphs and Combinatorics **18** (2002), 209–217.

[5] Francesco M Malvestuto, *A new notion of convexity in digraphs with an application to bayesian networks*, Discrete Mathematics, Algorithms and Applications **9** (2017), no. 02, 1750016.

[6] John L Pfaltz, *Convexity in directed graphs*, Journal of Combinatorial Theory, Series B **10** (1971), no. 2, 143–162.

[7] Gerard Sierksma, *Carathéodory and helly-numbers of convex-product-structures*, Pacific Journal of Mathematics **61** (1975), no. 1, 275–282.

[8] Marcel LJ van De Vel, *Theory of convex structures*, Elsevier, 1993.